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Trace techniques for angular momentum operators

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Abstract. General expressions for traces of products of angular momentum matrices (both in Cartesian and spherical bases) for systems of arbitrary spin are derived using the well established properties. It is shown that the trace is a polynomial in η where η is the eigenvalue of the J^2 operator. The trace techniques developed herein are applied to the problems of spin orientation.

1. Introduction

Ambler *et al* (1962a, b) have obtained the traces of products of a limited number of angular momentum matrices for systems of arbitrary spin. The purpose of this article is to extend the results of Ambler *et al* and to obtain the nature and the general form of the trace of a product of an arbitrary number of angular momentum matrices. In § 2 it is shown that the trace is a polynomial in η where η is the eigenvalue of the J^2 operator. We also prove that the trace can be expanded in terms of $\mathrm{Tr}(J_z^{2p})$, where p is a positive integer. The results are tabulated for a product of matrices up to eleven in number in a more concise form than given by Ambler *et al* (1962a).

The trace techniques developed here are of great use in the study of spin orientation problems. For the purpose of illustration we have obtained the explicit forms for the spin tensors and evaluated their reduced matrix elements (§ 3.2) and we have also investigated the elastic scattering of particles of arbitrary spin by a target nucleus of zero spin (§ 3.3).

In this paper, the following notation is used:

(i) Angular momenta are expressed in units of $\hbar$ and are denoted by j.

(ii) The symbols λ, μ, ν denote x or y or z.

(iii) The Levi-Civita symbol in three dimensions is denoted by $\epsilon_{\lambda\mu\nu}$.

(iv) Positive integers including zero are denoted by α, β, γ, n, p, q, r and K.

(v) The symbols L, M and N denote x, y and z in any permutation (L, M and N are different).

(vi) The symbols η and Ω are defined by $\eta = j(j+1)$, $\Omega = \eta(2j+1)$.

(vii) The difference operator (cf Miller 1961) is Δ and the anti-difference operator is Δ^{-1}.

(viii) The Bernoulli polynomial of the first kind (cf Miller 1961) of degree p in s is denoted by $B_p(s)$.

(ix) Polynomials of degree p in ξ with real and rational coefficients are denoted by $F_p(\xi)$, $G_p(\xi)$, ... A polynomial in ξ of degree -1 is interpreted to have the $1/\xi$ term only.

(x) The scalar triple product is denoted by

$$[\boldsymbol{PQR}] = (\boldsymbol{P}\times\boldsymbol{Q}) . \boldsymbol{R}.$$

2. Evaluation of the trace

2.1. Properties of angular momentum matrices and traces

In this section we give the well known properties of the angular momentum matrices and of the traces which are used to develop the techniques.

(i) If $\boldsymbol{J}$ is an angular momentum operator, then

$$\boldsymbol{J}\times\boldsymbol{J} = \mathrm{i}\boldsymbol{J}. \tag{2.1}$$

Equation (2.1) is a consequence of the more fundamental relation

$$J_\lambda J_\mu - J_\mu J_\lambda = \mathrm{i}\epsilon_{\lambda\mu\nu} J_\nu . \tag{2.2}$$

We have used the Einstein summation convention of summing over repeated indices in equation (2.2).

(ii) The scalar matrix $\boldsymbol{J}^2$ is given by

$$\boldsymbol{J}^2 = \eta \boldsymbol{I}, \tag{2.3}$$

where $\boldsymbol{I}$ is a unit matrix of order $(2j+1)$.

(iii) The components of $\boldsymbol{J}$ namely, J_x, J_y and J_z, are Hermitian.

(iv) The matrices $\boldsymbol{J}^2$ and J_λ can be diagonalized simultaneously in which case J_λ will have a spectrum of eigenvalues (diagonal elements) $-j$ to j in steps of unity. Of the two remaining matrices, one has only real elements and the other has only imaginary elements (cf Rose 1957a).

(v) The trace is invariant under a similarity transformation.

(vi) The trace of a product of matrices is not changed by a cyclic permutation of the matrices.

2.2. Nature of the trace

Consider a product of angular momentum matrices

$$A = J_\lambda J_\mu J_\nu \ldots . \tag{2.4}$$

Let J_L occur α times, J_M, β times and J_N, γ times†. We can show that Tr A is real when α, β and γ are all even and purely imaginary when α, β and γ are all odd. When α, β and γ are of mixed type (even and odd mixture), Tr A is identically zero, zero for all j. The proof is given below and it is based on the property that in a given representation, two of the three matrices consist of real elements whereas the third consists of purely imaginary elements.

Case I: α, β and γ are of mixed type.
Consider two different representations which are connected by a unitary transformation. In one representation, let the matrix which occurs an odd number of times have purely imaginary elements. Since the other two matrices have real elements, TrA is purely

† For any matrix $\mathscr{J}$, $\mathscr{J}^0 = I$, the unit matrix.

imaginary. In the other representation, let the matrix which occurs an even number of times have purely imaginary elements. In this representation Tr A is real. Since the trace is invariant under a similarity transformation,

$$\operatorname{Tr} A = 0. \tag{2.5}$$

Therefore, when α, β and γ are of mixed type, Tr A is identically zero, zero for all j.

Case II: α, β and γ are all odd (even).
Consider the representation in which any one of J_L, J_M and J_N has purely imaginary elements. Tr A is purely imaginary (real) as each matrix occurs an odd (even) number of times and i^{odd} (i^{even}) is purely imaginary (real).

Therefore, using again the property that the trace is invariant under a similarity transformation, Tr A is purely imaginary (real) when α, β and γ are all odd (even).

Thus, by mere inspection, one can tell whether the trace is real, zero (zero for all j), or purely imaginary.

2.3. *Evaluation of* $Tr(J_\lambda^{2p})$

The evaluation of $\operatorname{Tr}(J_\lambda^{2p})$ is straightforward. In the representation $|jm\rangle$ in which J_λ is diagonal,

$$\operatorname{Tr}(J_\lambda^{2p}) = \sum_{m=-j}^{j} \langle jm|J_\lambda^{2p}|jm\rangle = \sum_m m^{2p}. \tag{2.6}$$

Since (cf Miller 1961)

$$\Delta B_{2p+1}(m) = (2p+1)m^{2p}, \tag{2.7}$$

$$\sum_{m=-j}^{j} m^{2p} = \Delta^{-1} m^{2p}\Big|_{-j}^{j+1} = \frac{2}{2p+1} B_{2p+1}(j+1), \tag{2.8}$$

using the fundamental theorem of the sum calculus and the relation (cf Miller 1961)

$$B_{2p+1}(1-s) = -B_{2p+1}(s). \tag{2.9}$$

Hence

$$\operatorname{Tr}(J_\lambda^{2p}) = \frac{2}{2p+1} B_{2p+1}(j+1). \tag{2.10}$$

Using the principle of mathematical induction we can show that (cf Subramanian 1973)

$$B_{2p+1}(s) = s(s-1)(2s-1)F_{p-1}(u), \tag{2.11}$$

where $u = s^2 - s$. The proof involves the use of the following relations (cf Miller 1961):

(i) $$\int_a^s B_p(t)\,dt = \frac{1}{p+1}(B_{p+1}(s) - B_{p+1}(a)), \tag{2.12}$$

(ii) $$B_{2p+3}(1) = 0, \tag{2.13}$$

(iii) $$B_{2p+3}(\tfrac{1}{2}) = 0. \tag{2.14}$$

Hence

$$\mathrm{Tr}(J_\lambda^{2p}) = \frac{2\Omega}{2p+1} G_{p-1}(\eta). \tag{2.15}$$

Since the trace is invariant under a similarity transformation, the above result is true in all representations.

2.4. Results

Using equation (2.10) and the properties given in § 2.1 we have obtained the traces of products of matrices up to eleven in number. Table 1 gives a list of the trace of a product of angular momentum matrices in a Cartesian basis. Traces of other products can easily be obtained using table 1.

The following properties of the traces can be easily proved:

(*a*) $\mathrm{Tr}(J_L^\alpha J_M^\beta J_N^\gamma)$ is invariant under an interchange of the powers of J_L, J_M and J_N.

(*b*) $\mathrm{Tr}(J_L^\alpha J_M^\beta J_N^\gamma) = (-1)^{\alpha+\beta+\gamma} \mathrm{Tr}(J_L^\alpha J_N^\gamma J_M^\beta)$. (2.16)

In some problems the angular momentum matrices occur as scalar products $\boldsymbol{J} \cdot \boldsymbol{A}$ where A is a vector but not an angular momentum operator. Hence it is desirable to obtain traces of such scalar products using table 1 and they are presented in table 2.

2.5. General form of the trace of a product

In § 2.2 we have found the nature of the trace of a product of an arbitrary number of angular momentum matrices. In this subsection we obtain the general form of the trace of a product of angular momentum matrices given in a spherical as well as a Cartesian basis.

2.5.1. Product involving J_+, J_- and J_z. The raising and lowering operators are given by (cf Rose 1957a, chap 2)

$$J_\pm = J_x \pm \mathrm{i} J_y. \tag{2.17}$$

Consider a product of angular momentum matrices involving J_+, J_- and J_z,

$$B = J_a J_b J_c \ldots, \tag{2.18}$$

where each one of the matrices J_a, J_b, J_c, ... can be any one of J_+, J_- and J_z. Let J_+ occur p times, J_-, q times and J_z, r times.

$$\mathrm{Tr}\, B = \sum_m \langle jm|B|jm \rangle. \tag{2.19}$$

Since the trace is invariant under a similarity transformation, let us evaluate the trace in the representation in which J_z is diagonal. Since m is stepped up p times, stepped down q times, and unaffected r times,

$$m + p \times (1) + q \times (-1) + r \times (0) = m. \tag{2.20}$$

Table 1. Traces of products of angular momentum matrices in Cartesian basis

Number	A	$\mathrm{Tr}\, A$
1	J_L^2	$\frac{1}{3}\Omega$
2	$J_\lambda J_\mu J_\nu$	$\mathrm{i}\frac{1}{6}\Omega\epsilon_{\lambda\mu\nu}$
3	J_L^4	$\frac{1}{15}\Omega(3\eta-1)$
4	$J_L^2 J_M^2$	$\frac{1}{30}\Omega(2\eta+1)$
5	$J_\lambda^3 J_\mu J_\nu$	$\mathrm{i}\frac{1}{30}\Omega(3\eta-1)\epsilon_{\lambda\mu\nu}$
6	J_L^6	$\frac{1}{21}\Omega(3\eta^2-3\eta+1)$
7	$J_L^4 J_M^2$	$\frac{1}{210}\Omega(6\eta^2+8\eta-5)$
8	$J_L^2 J_M^2 J_N^2$	$\frac{1}{210}\Omega(2\eta^2-9\eta+10)$
9	$J_\lambda^5 J_\mu J_\nu$	$\mathrm{i}\frac{1}{42}\Omega(3\eta^2-3\eta+1)\epsilon_{\lambda\mu\nu}$
10	$J_\lambda^3 J_\mu^3 J_\nu$	$\mathrm{i}\frac{1}{420}\Omega(18\eta^2+3\eta-8)\epsilon_{\lambda\mu\nu}$
11	J_L^8	$\frac{1}{45}\Omega(5\eta^3-10\eta^2+9\eta-3)$
12	$J_L^6 J_M^2$	$\frac{1}{630}\Omega(10\eta^3+25\eta^2-48\eta+21)$
13	$J_L^4 J_M^4$	$\frac{1}{210}\Omega(2\eta^3+8\eta^2-7\eta+1)$
14	$J_L^4 J_M^2 J_N^2$	$\frac{1}{630}\Omega(2\eta^3-25\eta^2+54\eta-24)$
15	$J_\lambda^7 J_\mu J_\nu$	$\mathrm{i}\frac{1}{90}\Omega(5\eta^3-10\eta^2+9\eta-3)\epsilon_{\lambda\mu\nu}$
16	$J_\lambda^5 J_\mu^3 J_\nu$	$\mathrm{i}\frac{1}{420}\Omega(10\eta^3+10\eta^2-33\eta+16)\epsilon_{\lambda\mu\nu}$
17	$J_\lambda^3 J_\mu^3 J_\nu^3$	$\mathrm{i}\frac{1}{420}\Omega(6\eta^3-3\eta^2+27\eta-20)\epsilon_{\lambda\mu\nu}$
18	J_L^{10}	$\frac{1}{33}\Omega(3\eta^4-10\eta^3+17\eta^2-15\eta+5)$
19	$J_L^8 J_M^2$	$\frac{1}{990}\Omega(10\eta^4+40\eta^3-156\eta^2+192\eta-75)$
20	$J_L^6 J_M^4$	$\frac{1}{6930}\Omega(30\eta^4+230\eta^3-292\eta^2-51\eta+105)$
21	$J_L^6 J_M^2 J_N^2$	$\frac{1}{6930}\Omega(10\eta^4-235\eta^3+856\eta^2-1062\eta+420)$
22	$J_L^4 J_M^4 J_N^2$	$\frac{1}{6930}\Omega(6\eta^4-196\eta^3+353\eta^2+135\eta-210)$
23	$J_\lambda^9 J_\mu J_\nu$	$\mathrm{i}\frac{1}{66}\Omega(3\eta^4-10\eta^3+17\eta^2-15\eta+5)\epsilon_{\lambda\mu\nu}$
24	$J_\lambda^7 J_\mu^3 J_\nu$	$\mathrm{i}\frac{1}{1980}\Omega(30\eta^4+65\eta^3-358\eta^2+477\eta-192)\epsilon_{\lambda\mu\nu}$
25	$J_\lambda^5 J_\mu^5 J_\nu$	$\mathrm{i}\frac{1}{924}\Omega(10\eta^4+40\eta^3-123\eta^2+93\eta-20)\epsilon_{\lambda\mu\nu}$
26	$J_\lambda^5 J_\mu^3 J_\nu^3$	$\mathrm{i}\frac{1}{4620}\Omega(30\eta^4-45\eta^3+588\eta^2-1107\eta+512)\epsilon_{\lambda\mu\nu}$
27	J_L^{2p}	$\dfrac{2}{2p+1}B_{2p+1}(j+1)$
28	$J_L^p J_M^2$	$\frac{1}{2}[\eta\, \mathrm{Tr}(J_L^p)-\mathrm{Tr}(J_L^{p+2})]$
29	$J_\lambda^{2p+1} J_\mu J_\nu$	$\mathrm{i}\frac{1}{2}[\mathrm{Tr}(J_z^{2p+2})]\epsilon_{\lambda\mu\nu}$

Hence $p = q$, and therefore J_+ and J_- must occur the same number of times for the trace to exist. Since (cf Rose 1957a, chap 2)

$$J_\pm|j, m+P\rangle = [j\mp(m+P)]^{1/2}[j\pm(m+P)+1]^{1/2}|j, m+P\pm 1\rangle,$$
$$J_z|j, m+R\rangle = (m+R)|j, m+R\rangle, \tag{2.21}$$

the quantity $\langle jm|B|jm\rangle$ is a product of p products of the type $[(j-m-P)(j+m+P+1)]^{1/2}$, q products of the type $[(j+m+Q)(j-m-Q+1)]^{1/2}$ and r products of the type

Table 2. Traces of scalar products involving angular momentum operators

Number	Operator	Trace
1	$\boldsymbol{J.A}$	0
2	$\boldsymbol{J}(\boldsymbol{J.A})$	$\frac{1}{3}\Omega\boldsymbol{A}$
3	$(\boldsymbol{J.A})(\boldsymbol{J.B})$	$\frac{1}{3}\Omega(\boldsymbol{A.B})$
4	$\boldsymbol{J}(\boldsymbol{J.A})(\boldsymbol{J.B})$	$\mathrm{i}\frac{1}{6}\Omega(\boldsymbol{A}\times\boldsymbol{B})$
5	$J_\lambda J_\mu(\boldsymbol{J.A})$	$\mathrm{i}\frac{1}{6}\Omega\epsilon_{\lambda\mu\nu}A_\nu$
6	$(J_\lambda J_\mu + J_\mu J_\lambda)(\boldsymbol{J.A})$	0
7	$(\boldsymbol{J.A})(\boldsymbol{J.B})(\boldsymbol{J.C})$	$\mathrm{i}\frac{1}{6}\Omega[\boldsymbol{ABC}]$
8	$\boldsymbol{J}(\boldsymbol{J.A})(\boldsymbol{J.B})(\boldsymbol{J.C})$	$\frac{1}{30}\Omega[(2\eta+1)(\boldsymbol{A.B})\boldsymbol{C}+(2\eta+1)(\boldsymbol{B.C})\boldsymbol{A}+2(\eta-2)(\boldsymbol{C.A})\boldsymbol{B}]$
9	$(\boldsymbol{J.A})(\boldsymbol{J.B})(\boldsymbol{J.C})(\boldsymbol{J.D})$	$\frac{1}{30}\Omega[(2\eta+1)(\boldsymbol{A.B})(\boldsymbol{C.D})+(2\eta+1)(\boldsymbol{B.C})(\boldsymbol{D.A})+2(\eta-2)(\boldsymbol{C.A})(\boldsymbol{B.D})]$
10	$\boldsymbol{J}(\boldsymbol{J.A})(\boldsymbol{J.B})(\boldsymbol{J.C})(\boldsymbol{J.D})$	$\mathrm{i}\frac{1}{30}\Omega\{(3\eta-1)(\boldsymbol{A.B})(\boldsymbol{C}\times\boldsymbol{D})+(3\eta-1)(\boldsymbol{C.D})(\boldsymbol{A}\times\boldsymbol{B})$ $+(4\eta-3)(\boldsymbol{A.D})(\boldsymbol{B}\times\boldsymbol{C})+(\eta-2)[\boldsymbol{ABD}]\boldsymbol{C}-(\eta-2)[\boldsymbol{ABC}]\boldsymbol{D}\}$
11	$(\boldsymbol{J.A})(\boldsymbol{J.B})(\boldsymbol{J.C})(\boldsymbol{J.D})(\boldsymbol{J.E})$	$\mathrm{i}\frac{1}{30}\Omega\{(3\eta-1)(\boldsymbol{A.B})[\boldsymbol{CDE}]+(3\eta-1)(\boldsymbol{C.D})[\boldsymbol{ABE}]$ $+(4\eta-3)(\boldsymbol{A.D})[\boldsymbol{BCE}]+(\eta-2)(\boldsymbol{C.E})[\boldsymbol{ABD}]$ $-(\eta-2)(\boldsymbol{D.E})[\boldsymbol{ABC}]\}$

$(m+R)$ where P, Q and R are integers. Since $p = q$, and since

$$J_+|j, m+P\rangle = [(j-m-P)(j+m+P+1)]^{1/2}|j, m+P+1\rangle,$$
$$J_-|j, m+P+1\rangle = [(j+m+P+1)(j-m-P)]^{1/2}|j, m+P\rangle, \tag{2.22}$$

the contributions from J_+ and J_- can be paired to give a typical term $[\eta - m^2 - (2P+1)m - (P^2+P)]$ and contributions to $\langle jm|B|jm\rangle$ by J_+ and J_- through p such products are given by

$$\sum_{K=0}^{2p} F_d(\eta) m^{2p-K}$$

with $d = \frac{1}{2}K$ or $\frac{1}{2}(K-1)$ according to whether K is even or odd. Similarly the contribution to $\langle jm|B|jm\rangle$ by J_z matrices is given by

$$(m+R_1)(m+R_2)\dots(m+R_r) = \sum_{\rho=0}^{r} a_\rho m^\rho, \tag{2.23}$$

where $R_1, R_2, \dots, R_r$ and a_ρ are integers. Using equation (2.15), we get

$$\begin{aligned}\operatorname{Tr} B &= \sum_m \sum_{K=0}^{2p} \sum_{\rho=0}^{r} a_\rho F_d(\eta) m^{2p-K+\rho},\\ &= \sum_{K,\rho} a_\rho F_d(\eta) \operatorname{Tr}(J_z^{2p-K+\rho}),\\ &= \Omega F(\eta),\end{aligned} \tag{2.24}$$

where $F(\eta)$ is a polynomial in η with real and rational coefficients. The maximum degree of η in $F(\eta)$ comes from $\mathrm{Tr}(J_z^{2p+r})$ when r is even and from $\mathrm{Tr}(J_z^{2p+r-1})$ when r is odd, as $\mathrm{Tr}(J_z^{2\beta+1})$ is zero. We now obtain the following results.

Case I: r is even.

$$\mathrm{Tr}\, B = \Omega F_S(\eta), \tag{2.25}$$

with

$$S = \tfrac{1}{2}(2p+r-2). \tag{2.26}$$

Case II: r is odd.

$$\mathrm{Tr}\, B = \Omega G_T(\eta), \tag{2.27}$$

with

$$T = \tfrac{1}{2}(2p+r-3). \tag{2.28}$$

The above results are very general and applicable to a product of any number of matrices. It is a matter of simple algebra to show that the results of Ambler *et al* (1962b) given for the limited case $p+q+r \leqslant 9$ can be brought to the form given by equations (2.25)–(2.28).

2.5.2. Spherical basis: product involving J_1^1, J_1^{-1}, J_1^0. The angular momentum matrices J_1^1, J_1^{-1}, J_1^0 are given by

$$J_1^1 = -\frac{1}{\sqrt{2}}J_+, \qquad J_1^{-1} = \frac{1}{\sqrt{2}}J_-, \qquad J_1^0 = J_z. \tag{2.29}$$

Consider a product of angular momentum matrices

$$C = J_1^P J_1^Q J_1^R \ldots, \tag{2.30}$$

where P, Q, R, ... can be any one of 1, -1, 0. Let J_1^1 occur p times, J_1^{-1} q times and J_1^0, r times. It is easy to obtain

$$\mathrm{Tr}\, C = \Omega G(\eta)\delta_{pq}, \tag{2.31}$$

where $G(\eta)$ is a polynomial in η of degree S or T according to whether r is even or odd, with real and rational coefficients. The quantities S and T are given by equations (2.26) and (2.28) respectively.

2.5.3. Rectangular basis: product involving J_x, J_y *and* J_z. Let us now evaluate Tr A where A is given by equation (2.4). Since

$$J_x = \frac{J_+ + J_-}{2}, \qquad J_y = \frac{J_+ - J_-}{2\mathrm{i}} \tag{2.32}$$

$$\mathrm{Tr}\, A = (2)^{-\alpha}(2\mathrm{i})^{-\beta}\,\mathrm{Tr}[(J_+ + J_-)\ldots(J_+ - J_-)\ldots J_z \ldots], \tag{2.33}$$

and hence Tr A is a sum of $2^{\alpha+\beta}$ terms of the type $\mathrm{Tr}(J_+ \ldots J_- \ldots J_z \ldots)$. Let us evaluate Tr A in the representation in which J_z is diagonal. Let J_+ occur p times, and J_-, q times in a typical term; J_z occurs γ times. Since $p+q = \alpha+\beta$, and p and q must be equal for the trace to exist, we get $p = q = \frac{1}{2}(\alpha+\beta)$, and hence contributions to Tr A come only from such products which have $p = q = \frac{1}{2}(\alpha+\beta)$. Using equations (2.25)–(2.28) we obtain the following results.

Case I: α, β and γ are all even.

Since $(i)^{-\beta}$ is real,

$$\mathrm{Tr}\, A = \Omega F(\eta), \tag{2.34}$$

where $F(\eta)$ is a polynomial in η of degree S with real and rational coefficients and

$$S = \tfrac{1}{2}(\alpha+\beta+\gamma-2). \tag{2.35}$$

Case II: α, β and γ are all odd.

Since $(i)^{-\beta}$ is purely imaginary,

$$\mathrm{Tr}\, A = i\Omega G(\eta), \tag{2.36}$$

where $G(\eta)$ is a polynomial in η of degree T with real and rational coefficients and

$$T = \tfrac{1}{2}(\alpha+\beta+\gamma-3). \tag{2.37}$$

The results reported in table 1 of this paper and table 2 of Ambler *et al* (1962a) are in conformity with equations (2.34)–(2.37). Let us illustrate these equations with some examples.

$$\mathrm{Tr}(J_L^8 J_M^2 J_N^2) = \Omega \sum_{K=0}^{5} C_K \eta^K, \tag{2.38}$$

$$\mathrm{Tr}(J_\lambda^7 J_\mu^5 J_\nu^3) = i\Omega\left(\sum_{R=0}^{6} D_R \eta^R\right)\epsilon_{\lambda\mu\nu}. \tag{2.39}$$

To find C_K (D_R) we must know $\mathrm{Tr}(J_L^8 J_M^2 J_N^2)$ [$\mathrm{Tr}(J_\lambda^7 J_\mu^5 J_\nu^3)$] for six (seven) different values of j. Once C_K and D_R are known equations (2.38) and (2.39) can be used for all values of j.

2.6. Expansion of trace in terms of $Tr(J_\lambda^{2p})$

Now we prove that the trace of a product of angular momentum matrices given either in a spherical or a Catesian basis can be expanded in terms of $\mathrm{Tr}(J_\lambda^{2p})$. We have

$$F_p(\eta) = \sum_{K=0}^{p} A_K \eta^K G_{p-K}(\eta), \tag{2.40}$$

where the constants A_K are obtained by comparing the corresponding coefficients of η^K on both sides. Let the matrices B and A be given by equations (2.18) and (2.4). Using equations (2.15) and (2.40) we have the following results with $n = \alpha+\beta+\gamma$.

Case I: α, β and γ are all even.

$$\begin{aligned}\mathrm{Tr}\, B &= A_0\,\mathrm{Tr}(J_\lambda^n) + A_1\eta\,\mathrm{Tr}(J_\lambda^{n-2}) + A_2\eta^2\,\mathrm{Tr}(J_\lambda^{n-4}) + \dots, \\ &= \sum_{R=0}^{(n-2)/2} A_R \eta^R\,\mathrm{Tr}(J_\lambda^{n-2R}).\end{aligned} \tag{2.41}$$

Similarly

$$\mathrm{Tr}\, A = \sum_{R=0}^{(n-2)/2} B_R \eta^R\,\mathrm{Tr}(J_\lambda^{n-2R}). \tag{2.42}$$

Case II: α, β and γ are all odd.

$$\mathrm{Tr}\, B = \sum_{R=0}^{(n-3)/2} C_R \eta^R \,\mathrm{Tr}(J_\lambda^{n-1-2R}). \tag{2.43}$$

$$\mathrm{Tr}\, A = \mathrm{i} \sum_{R=0}^{(n-3)/2} D_R \eta^R \,\mathrm{Tr}(J_\lambda^{n-1-2R}). \tag{2.44}$$

In equations (2.41)–(2.44), A_R, B_R, C_R and D_R are real and rational numbers. Thus the traces of products of angular momentum matrices can be derived† from those of the form $\mathrm{Tr}(J_\lambda^{2p})$.

3. Applications

3.1. Anti-commutation relations for angular momentum matrices

The familiar result that the angular momentum matrices J_L, J_M cannot anti-commute when j is neither zero nor half can be obtained using the results given in table 1. Using table 1 we have

$$\mathrm{Tr}(J_L J_M J_L J_M) = \frac{\Omega}{15}(\eta - 2) \qquad (\textit{no}\ \text{summation}). \tag{3.1}$$

Hence

$$\mathrm{Tr}(J_L^2 J_M^2) + \mathrm{Tr}(J_L J_M J_L J_M) = \tfrac{1}{30}[j(j+1)(2j+1)(2j-1)(2j+3)] \qquad (\textit{no}\ \text{summation}). \tag{3.2}$$

Since j is necessarily positive, the right-hand side of equation (3.2) is zero only when $j = 0, \frac{1}{2}$. This then implies that J_L, J_M cannot anti-commute when j is neither zero nor half. If they do so, the left-hand side of equation (3.2) reduces to zero irrespective of the values of j.

3.2. Nuclear spin orientation

The density matrix ρ_f for the final nuclear state completely describes its spin orientation (cf Lakin 1955, Devanathan *et al* 1972, Ramachandran 1967) which can be represented conveniently by a set of parameters $\langle T_K^{m_K} \rangle$ defined by

$$\langle T_K^{m_K} \rangle = \frac{\mathrm{Tr}(T_K^{m_K} \rho_f)}{\mathrm{Tr}(\rho_f)}, \tag{3.3}$$

where $T_K^{m_K}$ denotes a spherical tensor operator of rank K in the spin space of the final nucleus and satisfies the normalization condition

$$\mathrm{Tr}(T_K^{m_K^\dagger} T_{K'}^{m_{K'}}) = (2j+1)\delta_{K,K'}\delta_{m_K,m_{K'}}, \tag{3.4}$$

subject to the restriction $0 \leqslant K \leqslant 2j$, where j is the spin of the final state.

Using trace techniques, we show below that the following relations‡

(i) $$\langle j \| T_K \| j \rangle = (2K+1)^{1/2}, \tag{3.5}$$

† A statement made without proof by Ambler *et al* (1962a, b).

‡ For angular momentum coefficients and reduced matrix elements, we follow the notation and definition of Rose (1957a).

(ii) $$T_K^K = \frac{(-1)^K}{K!}\left(\frac{(2j+1)(2K+1)!(2j-K)!}{(2j+1+K)!}\right)^{1/2}(J_+)^K, \tag{3.6}$$

can be obtained†.

From the Wigner–Eckart theorem

$$\langle j'm'|T_K^{m_K}|jm\rangle = C(jKj';mm_Km')\langle j'\|T_K\|j\rangle, \tag{3.7}$$

and $m' = m+m_K$. From the fact that the commutator $[J^2, T_K^{m_K}] = 0$, we see that $j = j'$. Hence

$$T_{K'}^{m_{K'}}|jm\rangle = C(jK'j;mm_{K'})\langle j\|T_{K'}\|j\rangle|j, m+m_{K'}\rangle, \tag{3.8}$$

$$\mathrm{Tr}(T_K^{m_K^+}T_{K'}^{m_{K'}}) = \sum_m C(jK'j;mm_{K'})\langle j\|T_{K'}\|j\rangle(-1)^{m_K}C(jKj;m+m_{K'},-m_K,m)\langle j\|T_K\|j\rangle. \tag{3.9}$$

Using the symmetry and orthogonal properties of the C coefficients (Rose 1957a, chap 3), we get

$$\mathrm{Tr}(T_K^{m_K^+}T_{K'}^{m_{K'}}) = \frac{(2j+1)}{(2K+1)}(\langle j\|T_K\|j\rangle)^2\delta_{K,K'}\delta_{m_K,m_{K'}}. \tag{3.10}$$

From equations (3.4) and (3.10) we get equation (3.5), assuming the reduced matrix element to be real and positive.

To obtain T_K^K, we proceed as follows. Let us construct the spin tensors using angular momentum operator $\boldsymbol{J}$. Thus

$$T_K^K = G_K(J_1^1)^K = G_K\left(-\frac{1}{\sqrt{2}}\right)^K(J_+)^K, \tag{3.11}$$

where G_K is a constant depending upon K. Now

$$\begin{aligned}\langle jj|T_K^K|j, j-K\rangle &= \langle j\|T_K\|j\rangle C(jKj;j-K,K)\\ &= \frac{(-1)^K}{K!}\left(\frac{(2j+1)!(2K+1)!K!}{(2j+1+K)!}\right)^{1/2},\end{aligned} \tag{3.12}$$

using the Wigner–Eckart theorem, equation (3.5) and the expression given by Racah (1942) for the C coefficient.

Using equations (3.11) and (2.21) we also have

$$\langle jj|T_K^K|j, j-K\rangle = G_K\left(-\frac{1}{\sqrt{2}}\right)^K\left(\frac{K!(2j)!}{(2j-K)!}\right)^{1/2}. \tag{3.13}$$

From equations (3.12) and (3.13) we have

$$G_K = \frac{1}{K!}\left(\frac{2^K(2j+1)(2K+1)!(2j-K)!}{(2j+1+K)!}\right)^{1/2}. \tag{3.14}$$

Equation (3.6) follows easily from equations (3.11) and (3.14).

Incidentally, using equations (3.11), (3.4) and (3.14) we have

$$\mathrm{Tr}(J_-^KJ_+^K) = \frac{(2j+1+K)!(K!)^2}{(2K+1)!(2j-K)!}. \tag{3.15}$$

† Rose (1957b) and Goldfarb (1958) have obtained similar results using a different approach.

Equation (3.15) is true for any K and it agrees with the results of Ambler *et al* (1962b) given for limited values of $K = 1, 2, 3$ and 4.

The matrix element for $T_K^{m_K}$ can easily be found using equation (3.5) and the Wigner–Eckart theorem.

$$\langle jm'|T_K^{m_K}|jm\rangle = C(jKj; mm_Km')\langle j\|T_K\|j\rangle = (2K+1)^{1/2}C(jKj; mm_Km'). \tag{3.16}$$

We have obtained T_K^K using trace techniques. The same techniques can be used to find any $T_K^{m_K}$. For the sake of illustration we have obtained T_2^1 in the appendix.

3.3. *Elastic scattering of particles of arbitrary spin*

Let us consider the elastic scattering of particles of arbitrary spin j by a target nucleus of zero spin. The operator that is responsible for this can be of the general form

$$t = (\boldsymbol{J}.\boldsymbol{A})^{2j} + (\boldsymbol{J}.\boldsymbol{B})^{2j-1} + (\boldsymbol{J}.\boldsymbol{C})^{2j-2} + \dots, \tag{3.17}$$

since a tensor of maximum rank $2j$ is necessary to connect one projection of j to another projection of j.

The density matrix ρ_f for the scattered beam completely describes its spin orientation which can be represented conveniently by a set of parameters $\langle T_K^{m_K}\rangle$ defined by equation (3.3). $T_K^{m_K}$ denotes a spherical tensor operator of rank K in the spin space of the scattered beam satisfying the normalization condition equation (3.4). The differential cross section is given by Tr ρ_f where

$$\rho_f = tt^+. \tag{3.18}$$

Using table 2 we get the following relations for the special case

$$t = \boldsymbol{J}.\boldsymbol{C} + D. \tag{3.19}$$

(i) $$\mathrm{Tr}(\rho_f) = \frac{2j+1}{3}(\eta\boldsymbol{C}.\boldsymbol{C}^* + 3DD^*), \tag{3.20}$$

(ii) $$\mathrm{Tr}(\boldsymbol{T}_1\rho_f) = \left(\frac{3}{\eta}\right)^{1/2}\mathrm{Tr}(\boldsymbol{J}\rho_f),$$

$$= \left(\frac{\eta}{12}\right)^{1/2}(2j+1)[\mathrm{i}(\boldsymbol{C}\times\boldsymbol{C}^*) + 2\boldsymbol{C}D^* + 2D\boldsymbol{C}^*] \tag{3.21}$$

(iii) $$\mathrm{Tr}(T_2^0\rho_f) = \left(\frac{(4\eta-3)\eta}{180}\right)^{1/2}(2j+1)(3C_zC_z^* - \boldsymbol{C}.\boldsymbol{C}^*), \tag{3.22}$$

(iv) $$\mathrm{Tr}(T_2^{\pm1}\rho_f) = \mp\left(\frac{(4\eta-3)\eta}{120}\right)^{1/2}(2j+1) \times[(C_xC_z^* + C_zC_x^*) \pm \mathrm{i}(C_yC_z^* + C_zC_y^*)] \tag{3.23}$$

(v) $$\mathrm{Tr}(T_2^{\pm}2\rho_f) = \left(\frac{(4\eta-3)\eta}{120}\right)^{1/2}(2j+1) \times[(C_xC_x^* - C_yC_y^*) \pm \mathrm{i}(C_xC_y^* + C_yC_x^*)] \tag{3.24}$$

(vi) $$\mathrm{Tr}(T_K^{m_K}\rho_f) = 0, \qquad \text{for } K \geqslant 3. \tag{3.25}$$

In the above C^*, D^* denote complex conjugates of C and D respectively. Equation (3.25) is not surprising since the special choice of the operator given by equation (3.19) restricts the density matrix

$$\rho_f = tt^+ = (\boldsymbol{J}\,.\,\boldsymbol{C}+D)[(\boldsymbol{J}\,.\,\boldsymbol{C}+D)^+], \tag{3.26}$$

to contain only tensors of rank two and less. The normalization condition equation (3.4) clearly tells that the trace vanishes when $K \neq K'$.

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Appendix. Construction of $T_K^{m_K}$

In the text we have obtained T_K^K. The trace techniques can be used to find any $T_K^{m_K}$. For the sake of illustration let us obtain T_2^1. We have

$$\begin{aligned}(J_1 \times J_1)_2^1 &= C(112;01)J_1^0 J_1^1 + C(112;10)J_1^1 J_1^0 \\ &= -\tfrac{1}{2}(J_z J_+ + J_+ J_z),\end{aligned} \tag{A.1}$$

using the value of the C coefficients. Hence

$$T_2^1 = -\tfrac{1}{2}k(J_z J_+ + J_+ J_z), \tag{A.2}$$

where k is a positive constant. Using the normalization condition

$$\mathrm{Tr}(T_2^{1\,+} T_2^1) = 2j+1, \tag{A.3}$$

and evaluating the trace using table 1, k is found as

$$k = \left(\frac{30}{j(j+1)(2j-1)(2j+3)}\right)^{1/2}, \tag{A.4}$$

so that

$$T_2^1 = -\left(\frac{15}{2j(j+1)(2j-1)(2j+3)}\right)^{1/2}(J_z J_+ + J_+ J_z). \tag{A.5}$$

When $j = 1$,

$$T_2^1 = -\frac{\sqrt{3}}{2}(J_z J_+ + J_+ J_z). \tag{A.6}$$

Equation (A.6) is essentially the same as equation (1.3) of Lakin (1955). Only the notations used are different.

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